THE INEQUALITIES FOR SOME TYPES OF q-INTEGRALS

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Abstract. We discuss the inequalities for q-integrals because of the fact that the inequalities can be very useful in the future mathematical research. Since q-integral of a function over an interval $[a, b]$ is defined by the difference of two infinite sums, there a lot of unexpected troubles in analyzing analogs of well-known integral inequalities. In this paper, we will signify to some directions to exceed this problem.

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1 Introduction

The integral inequalities can be used for the study of qualitative and quantitative properties of integrals. In order to generalize and spread the existing inequalities, we specify two ways to overcome the problems which ensue from the general definition of q-integral. The first one is the restriction of the q-integral over $[a, b]$ to a finite sum (see [2]). The second one is indicated in [6] and it means introduction the definition of the q-integral of the Riemann type. At the start sections, we give all definitions of the q-integrals, their correlations and properties. In the other sections, we elaborate the q-analogues of the well–known inequalities in the integral calculus, as Chebyshev, Grüss, Hermite-Hadamard for all the types of the q-integrals. At last, we give a few new inequalities which are valid only for some types of the q-integrals.

In the fundamental books about q-calculus (for example, see [3] and [4]), the q-integral of the function f over the interval $[0, b]$ is defined by

$$I_q(f; 0, b) = \int_0^b f(x) d_q x = b(1-q) \sum_{n=0}^{\infty} f(bq^n) q^n \quad (0 < q < 1). \tag{1}$$

If f is integrable over $[0, b]$, then

$$\lim_{q \nearrow 1} I_q(f; 0, b) = \int_0^b f(x)\ dx = I(f; 0, b).$$

Generally accepted definition for q-integral over an interval $[a, b]$ is

$$I_q(f; a, b) = \int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x \quad (0 < q < 1). \tag{2}$$

The values of such defined q-integrals of the polynomials have very similar form to those in the standard integral calculus. So, for example, we it is valid

$$\int_a^b x^n d_q x = \frac{b^{n+1} - a^{n+1}}{[n+1]_q},$$

where

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1}, \qquad n \in \mathbb{N}.$$

But, the problems come when the integrand f is defined in $[a, b]$ and it is not defined in $[0, a]$. Obviously this definition cannot be applied on evaluating of the integrals of the form

$$\int_2^3 \ln(x - 1) d_q x.$$

2 The q-integrals and correlations

Let a, b and q be some real numbers such that $0 < a < b$ and $q \in (0, 1)$.

Beside the q-integrals defined by (1) and (2) we will consider two other types of the q-integrals.

In the paper [2], H. Gauchman has introduced *the restricted q-integral*

$$G_q(f; a, b) = \int_a^b f(x) \ d_q^G x = b(1 - q) \sum_{k=0}^{n-1} f(bq^k) q^k \quad (a = bq^n). \tag{3}$$

Let us notice that lower bound of integral is $a = bq^n$, i.e. it is tied by chosen b, q and positive integer n.

In the paper [6], we have introduced *Riemann-type q-integral* by

$$R_q(f; a, b) = \int_a^b f(t) d_q^R t = (b - a)(1 - q) \sum_{k=0}^{\infty} f\big(a + (b - a)q^k\big) q^k. \tag{4}$$

This definition includes only point inside the interval of the integration.

The different types of the q-integral defined by (1)–(4) can be denoted in the unique way by $J_q(\,\cdot\,; a_{(J)}, b)$, where J can be G, I or R. Interval of the integration $E_{(J)} = [a_{(J)}, b]$ of q-integral $J_q(\,\cdot\,; a_{(J)}, b)$ depends on its type:

$a_{(G)} = bq^n$, $n \in \mathbb{N}$, for $G_q(\,\cdot\,; a, b)$;
$a_{(I)} = 0$, for $I_q(\,\cdot\,; 0, b)$;
$a_{(I)} \in [0, b]$, for $I_q(\,\cdot\,; a, b)$;
$a_{(R)} \in [0, b]$, for $R_q(\,\cdot\,; a, b)$.

We can say that a real function f is *q-integrable* on $[0, b]$ or $[a, b]$ if the series in (1) and (2) converge. In the similar way, we say that f is *qR-integrable* on $[a, b]$ if the series in (4) converges.

From now on, it will be assumed that the function f is q-integrable on $[0, b]$ (qR-integrable on $[a, b]$) whenever $I_q(f; 0, b)$ or $I_q(f; a, b)$ $(R_q(f; a, b))$ appears in the formula.

In this research it is convenient to define the operators

$$\begin{aligned}
\widehat{\ } : f \mapsto \widehat{f}, \qquad & \widehat{f}(x) = f\big(a + (b-a)x\big), \\
\widetilde{\ } : f \mapsto \widetilde{f}, \qquad & \widetilde{f}(x) = bf(bx) - af(ax), \\
\breve{\ } : f \mapsto \breve{f}, \qquad & \breve{f}(x) = f(bx) - f(ax),
\end{aligned}$$

such that associate the functions defined on $[0, 1]$ to the function defined on $[a, b]$. Notice that, for $x \in [0, 1]$, it is

$$\widehat{(fg)}(x) = \widehat{f}(x)\ \widehat{g}(x), \quad \widetilde{(fg)}(x) = \frac{1}{b-a}\big(\widetilde{f}(x)\widetilde{g}(x) - ab\ \breve{f}(x)\breve{g}(x)\big). \tag{5}$$

The correlations between the q-integrals defined by (1)–(4) are given in the following lemma.

Lemma 2.1. *If the real function f is q-integrable on $[0, b]$ or qR-integrable on $[a, b]$ $(0 < a < b)$, then it holds*

$$\begin{aligned}
I_q(f; 0, b) &= \lim_{n\to\infty} G_q(f; bq^n, b), & (6)\\
I_q(f; a, b) &= I_q(\widetilde{f}; 0, 1), \quad \textit{where} \quad \widetilde{f}(x) = bf(bx) - af(ax), & (7)\\
R_q(f; a, b) &= (b-a) I_q(\widehat{f}; 0, 1), \quad \textit{where} \quad \widehat{f}(x) = f(a + (b-a)x), & (8)
\end{aligned}$$

Proof. Since $G_q(f; bq^n, b)$ $(n \in \mathbb{N})$ is the partial sum of the series $I_q(f; 0, b)$, the relation (6) is evident.

The equalities (7) and (8) are valid because of

$$I_q(f; a, b) = (1-q)\sum_{k=0}^{\infty}\big(bf(bq^k) - af(aq^k)\big)q^k = I_q(\widetilde{f}; 0, 1)$$

and

$$R_q(f; a, b) = (b-a)(1-q)\sum_{k=0}^{\infty} f(a + (b-a)q^k)q^k = (b-a)I_q(\widehat{f}; 0, 1). \ \Box$$

The mentioned connections can be used to derive the inequalities for all types of the q-integrals. By (6), the inequalities for the infinite sum $I_q(f; 0, b)$ can be derived in the limit process from this ones for $G_q(f; a, b)$ which are defined by the finite sum. Using (7) and (8), the integrals $I_q(f; a, b)$ and $R_q(f; a, b)$ can be considered as the q-integrals over $[0, 1]$. Nevertheless, the results for $I_q(f; a, b)$

are quite rough because the points outside of the interval of integration (i.e. points on $[0, a]$) are included.

According to (5) and Lemma 2.1, the following integral relations are valid:

$$R_q(fg; a, b) = (b - a) I_q\big(\widehat{(fg)}; 0, 1\big) = (b - a) I_q\big(\widehat{f}\,\widehat{g}; 0, 1\big), \tag{9}$$

$$I_q(fg; a, b) = I_q\big(\widetilde{(fg)}; 0, 1\big) = \frac{1}{b - a}\Big(I_q\big(\widetilde{f}\,\widetilde{g}; 0, 1\big) - ab\ I_q\big(\breve{f}\,\breve{g}; 0, 1\big)\Big). \tag{10}$$

3 q-Chebyshev inequality

In this section we give the q-analogues of Chebyshev inequality for the monotonic functions (see [5], pp. 239.). The discrete case of this inequality is used in [2] for the restricted q-integrals. We derive its variants for the rest of the q-integrals.

The function $f : [a, b] \to \mathbb{R}$ is called *q-increasing* (q-decreasing) on $[a, b]$ if $f(qx) \le f(x)$ ($f(qx) \ge f(x)$) whenever $x, qx \in [a, b]$. It is easy to see that if the function f is increasing (decreasing), then it is q-increasing (q-decreasing) too.

Theorem 3.1. *Let $f, g : E_{(J)} \to \mathbb{R}$ be two real functions, both q-decreasing or both q-increasing. If $J_q(\,\cdot\,; a_{(J)}, b)$ is the q-integral defined by* (1), (3) *or* (4), *it holds*

$$J_q(fg; a_{(J)}, b) \ge \frac{1}{b - a_{(J)}} J_q(f; a_{(J)}, b)\ J_q(g; a_{(J)}, b).$$

Proof. For $J_q(\,\cdot\,; a_{(J)}, b) = G_q(\,\cdot\,; a, b)$, $a = bq^n$, the inequality is proven in [2]. So, the inequalities

$$G_q(fg; bq^n, b) \ge \frac{1}{b - bq^n} G_q(f; bq^n, b)\ G_q(g; bq^n, b)$$

are valid for all $n = 1, 2, \ldots$. When $n \to \infty$, using (6) we get the desired inequality for $J_q(\,\cdot\,; a_{(J)}, b) = I_q(\,\cdot\,; 0, b)$. In the case $J_q(\,\cdot\,; a_{(J)}, b) = R_q(\,\cdot\,; a, b)$, from the q-monotonicity of the functions f and g on $[a, b]$ follows the q-monotonicity of the functions $\widehat{f}$ and $\widehat{g}$ on $[0, 1]$. Hence, we have

$$I_q(\widehat{f}\,\widehat{g}; 0, 1) \ge I_q(\widehat{f}; 0, 1)\ I_q(\widehat{g}; 0, 1).$$

According to (7) and (8) we get the required inequality. $\square$

The Chebyshev inequality in the source form is not valid for $I_q(\,\cdot\,; a, b)$, where $0 < a < b$.

Example 3.1 For $f(x) = x^3$ and $g(x) = x^4$ on the interval $[1, 2]$ we have

$$I_q(x^3 \cdot x^4; 1, 2) - I_q(x^3; 1, 2) I_q(x^4; 1, 2) = 255 \frac{1 - q}{1 - q^8} - 465 \frac{(1 - q)^2}{(1 - q^4)(1 - q^5)},$$

wherefrom we conclude that the inequality holds only for $q > 1/2$, but it has opposite sign for $q < 1/2$.

Lemma 3.2. *Let the function $f : [0, b] \to \mathbb{R}$ be increasing and $0 < a < b$. If there exist two positive constants l and L such that $a^2/b^2 \le l/L$ and for every $x, y \in [0, b]$ the inequality*

$$l \le \frac{f(x) - f(y)}{x - y} \le L$$

is valid, then the function $\widetilde{f} : [0, 1] \to \mathbb{R}$ is increasing too.

Proof. Under the conditions of the Lemma, for every $0 \le x < y \le b$ we have

$$l(y - x) \le f(y) - f(x) \le L(y - x).$$

Then it holds

$$\begin{aligned}\widetilde{f}(y) - \widetilde{f}(x) &= b\big(f(by) - f(bx)\big) - a\big(f(ay) - f(ax)\big)\\ &\ge (b^2 l - a^2 L)(y - x) \ge 0. \ \square\end{aligned}$$

Theorem 3.3. *Let $f, g : [0, b] \to \mathbb{R}$ be two real increasing functions. If there exist the constants l_f, L_f, l_g and L_g such that $a^2/b^2 \le l_f/L_f$, $a^2/b^2 \le l_g/L_g$ and*

$$l_f \le \frac{f(x) - f(y)}{x - y} \le L_f, \quad l_g \le \frac{g(x) - g(y)}{x - y} \le L_g$$

holds, then the inequalities are valid:

$$(a) \quad I_q(fg; a, b) \ge \frac{1}{b - a} I_q(f; a, b) I_q(g; a, b) - \frac{ab(b - a)}{[3]_q} L_f L_g$$

$$(b) \quad I_q(fg; a, b) \ge \frac{1}{b - a} I_q(f; a, b) I_q(g; a, b) - \frac{ab}{b - a}\big(f(b) - f(0)\big)\big(g(b) - g(0)\big).$$

Proof. Suppose that f and g are both increasing on $[0, b]$. Then, according to Lemma 3.2, $\widetilde{f}$ and $\widetilde{g}$ are both increasing and hence q-increasing on $[0, 1]$. With respect to (10) we can write

$$I_q(fg; a, b) = \frac{1}{b - a}\Big(I_q(\widetilde{f}\,\widetilde{g}; 0, 1) - ab\ I_q(\breve{f}\,\breve{g}; 0, 1)\Big).$$

Using Theorem 3.1, we have

$$I_q(\widetilde{f}\,\widetilde{g}; 0, 1) \ge I_q(\widetilde{f}; 0, 1)\ I_q(\widetilde{g}; 0, 1),$$

wherefrom

$$I_q(fg; a, b) \ge \frac{1}{b - a}\Big(I_q(f; a, b)\ I_q(g; a, b) - ab I_q(\breve{f}\,\breve{g}; 0, 1)\Big). \tag{11}$$

(a) Under the conditions satisfied by the functions f and g on $[0, b]$, it holds

$$\begin{aligned}I_q(\breve{f}\,\breve{g}; 0, 1) &= (1 - q)\sum_{k=0}^{\infty}\big(f(bq^k) - f(aq^k)\big)\big(g(bq^k) - g(aq^k)\big)q^k\\ &\le (1 - q)\sum_{k=0}^{\infty} L_f L_g (bq^k - aq^k)^2 q^k = L_f L_g (b - a)^2 \frac{1 - q}{1 - q^3}\end{aligned}$$

Substituting this estimation in (11), we get the first inequality.
(b) Since the functions f and g are increasing on $[0,b]$, it holds

$$I_q(\check{f}\,\check{g};0,1) \le (1-q)\big(f(b)-f(0)\big)\big(g(b)-g(0)\big)\sum_{k=0}^{\infty} q^k = \big(f(b)-f(0)\big)\big(g(b)-g(0)\big),$$

what with (11) gives the second inequality. $\square$

4 q-Grüss inequality

The Grüss inequality (see [5], pp. 296) can be understood as conversion of Chebyshev one.

Theorem 4.1. *Let $f,g : E_{(J)} \to \mathbb{R}$ be two real functions, such that $m \le f(x) \le M$, $\varphi \le g(x) \le \Phi$ on $E_{(J)}$, where m, M, φ, Φ are given real constants. If $J_q(\,\cdot\,; a_{(J)}, b)$ is the q-integral defined by* (1), (3) *or* (4), *it holds*

$$\left| \frac{1}{b-a_{(J)}} J_q(fg; a_{(J)}, b) - \frac{1}{\big(b-a_{(J)}\big)^2} J_q(f; a_{(J)}, b)\ J_q(g; a_{(J)}, b) \right| \le \frac{1}{4}(M-m)(\Phi-\varphi).$$

Proof. For the restricted q-integrals $G_q(\,\cdot\,; bq^n, b)$, the inequality is proven in [2]. So, for any arbitrary positive integer n, the inequality

$$\left| \frac{1}{b-bq^n} G_q(fg; bq^n, b) - \frac{1}{(b-bq^n)^2} G_q(f; bq^n, b)\ G_q(g; bq^n, b) \right| \le \frac{1}{4}(M-m)(\Phi-\varphi)$$

is valid. When $n \to \infty$, we get the required inequality for $I_q(\,\cdot\,; 0, b)$ via (6). Finally, providing the conditions of the theorem, the functions $\widehat{f}$ and $\widehat{g}$ are bounded on $[0,1]$ by the constants m, M, φ, Φ respective. Then,

$$\left| I_q(\widehat{f}\,\widehat{g}; 0,1) - I_q(\widehat{f}; 0,1)\ I_q(\widehat{g}; 0,1) \right| \le \frac{1}{4}(M-m)(\Phi-\varphi)$$

holds and using the relation (8), we get the inequality for $R_q(\,\cdot\,; a, b)$. $\square$

Example 4.1 For $f(x) = x$ and $g(x) = x^2$ on the interval $[1,2]$ we have

$$I_q(x\cdot x^2; 1,2) - I_q(x;1,2) I_q(x^2;1,2) = (1-2q)\frac{3\ (2-q)}{(1+q)(1+q^2)(1+q+q^2)}.$$

Including the boundaries of the functions $f(x)$ and $g(x)$, we can see that the formula of Grüss inequality will not be hold on for $q \in (0, 1/3)$.

Theorem 4.2. *Let $f, g : [0,b] \to \mathbb{R}$ be two bounded such that $m \le f(x) \le M$, $\varphi \le g(x) \le \Phi$ on $[0,b]$, where m, M, φ, Φ are given real constants. Then it holds*

$$\Big|\ \frac{1}{b-a} I_q(fg;a,b) - \frac{1}{(b-a)^2} I_q(f;a,b) I_q(g;a,b)\ \Big| \le \frac{1}{4}(M-m)(\Phi-\varphi)\left(1+\frac{4ab}{(b-a)^2}\right).$$

Proof. Having in mind the boundaries of f and g on $[0,b]$, we have

$$bm - aM \le \widetilde{f}(x) \le bM - am, \qquad b\varphi - a\Phi \le \widetilde{g}(x) \le b\Phi - a\varphi,$$

where $\widetilde{f}$ and $\widetilde{g}$ are the function defined on $[0,1]$. According to Theorem 4.1, we have

$$\Big|\ I_q(\widetilde{f}\ \widetilde{g};0,1) - I_q(\widetilde{f};0,1)\ I_q(\widetilde{g};0,1)\ \Big| \le \frac{1}{4}(bM - am - bm + aM)(b\Phi - a\varphi - b\varphi + a\Phi).$$

By using (10), we obtain

$$\begin{aligned}\Big|\ (b-a)I_q(fg;a,b) - I_q(f;a,b)\ I_q(g;a,b)\ \Big| - ab\ \Big|\ I_q(\breve{f}\ \breve{g};0,1)\ \Big| \\ \le \Big|\ (b-a)I_q(fg;a,b) - I_q(f;a,b)\ I_q(g;a,b) + ab\ I_q(\breve{f}\ \breve{g};0,1)\ \Big| \\ \le \tfrac{1}{4}(b-a)^2(M-m)(\Phi-\varphi).\end{aligned}$$

With respect to the boundaries of f and g on $[0,b]$, the estimation

$$\Big|\ I_q(\breve{f}\ \breve{g};0,1)\ \Big| \le (M-m)(\Phi-\varphi)$$

holds, what, finally, proves the statement. □

5 q-Hermite–Hadamard inequality

The Hermite–Hadamard inequality (see [5], pp. 10) is related to the Jensen inequality for the convex function. In [2] there is proved a variant of its analogue for the restricted q-integrals. Here we will formulate and prove another variant of the q-Hermite–Hadamard inequality for the restricted q-integrals and for the other types of q-integrals.

Theorem 5.1. *Let $f : [a,b] \to \mathbb{R}$ $(a = bq^n)$ be a convex function. Then it holds*

$$f\left(\frac{a+b}{[2]_q}\right) \le \frac{1}{b-a} G_q(f;a,b) \le \frac{1}{[2]_q}\left(q\ f\left(\frac{a}{q}\right) + f(b)\right).$$

Proof. According to the definition of the restricted q-integral, we have

$$\frac{1}{b-a}G_q(f;a,b)=\frac{1-q}{1-q^n}\sum_{k=0}^{n-1}f(bq^k)q^k=\left(\sum_{k=0}^{n-1}q^k\right)^{-1}\left(\sum_{k=0}^{n-1}f(bq^k)q^k\right).$$

If we assign

$$\overline{x}=\left(\sum_{k=0}^{n-1}\ q^k\right)^{-1}\left(\sum_{k=0}^{n-1}bq^k\ q^k\right)=\frac{b(1+q^n)}{1+q}=\frac{a+b}{1+q}$$

and apply Jensen inequality for the convex functions on the last term, we obtain

$$\frac{1}{b-a}G_q(f;a,b)\geq f(\overline{x})=f\left(\frac{a+b}{1+q}\right).$$

On the other side, using a variant of the reverse Jensen inequality (see [5], pp. 9.), we get

$$\begin{aligned}\frac{1}{b-a}G_q(f;a,b)&\leq\frac{b-\overline{x}}{b-bq^{n-1}}\ f(bq^{n-1})+\frac{\overline{x}-bq^{n-1}}{b-bq^{n-1}}\ f(b)\\&=\left(b-\frac{a}{q}\right)^{-1}\left(\left(b-\frac{a+b}{1+q}\right)f\left(\frac{a}{q}\right)+\left(\frac{a+b}{1+q}-\frac{a}{q}\right)f(b)\right)\\&=\frac{1}{1+q}\left(q\ f\left(\frac{a}{q}\right)+f(b)\right).\ \square\end{aligned}$$

Theorem 5.2. *Let $f:[0,b]\to\mathbb{R}$ be a continuous convex function. Then,*

$$f\left(\frac{b}{[2]_q}\right)\leq\frac{1}{b}I_q(f;0,b)\leq\frac{1}{[2]_q}\Big(q\ f(0)+f(b)\Big).$$

Proof. Since the function f satisfies the conditions of Theorem 5.1 on the intervals $[bq^n,b]$ for every $n\in\mathbb{N}$, the inequalities

$$f\left(\frac{bq^n+b}{[2]_q}\right)\leq\frac{1}{b-bq^n}G_q(f;bq^n,b)\leq\frac{1}{[2]_q}\left(q\ f\left(\frac{bq^n}{q}\right)+f(b)\right)$$

are valid. When $n\to\infty$, we obtain the desired inequality because f is continuous and (6) is satisfied. $\square$

Theorem 5.3. *Let $f:[a,b]\to\mathbb{R}$ be a continuous convex function. Then,*

$$f\left(\frac{aq+b}{[2]_q}\right)\leq\frac{1}{b-a}R_q(f;a,b))\leq\frac{1}{[2]_q}\Big(q\ f(a)+f(b)\Big).$$

Proof. Under the conditions which are satisfied by the function f on $[a,b]$, the function $\widehat{f}(x)=f\big(a+(b-a)x\big)$ satisfies the conditions of the Theorem 5.2 on $[0,1]$. Hence

$$\widehat{f}\left(\frac{1}{[2]_q}\right)\leq I_q(\widehat{f};0,1)\leq\frac{1}{[2]_q}\Big(q\ \widehat{f}(0)+\widehat{f}(1)\Big).$$

According to (9) and the continuity of the function f, we get the desired inequality. $\square$

Let us remember that the function f is convex on $[0, b]$ if for all $x, y \in [0, b]$ and $p_1 + p_2 > 0$

$$f\left(\frac{p_1 x + p_2 y}{p_1 + p_2}\right) \leq \frac{p_1 f(x) + p_2 f(y)}{p_1 + p_2}$$

holds. The convexity of the function $\widetilde{f}$ on $[0, 1]$ is due to the existence of the appropriate constants l and L such that the condition

$$l \leq \frac{p_1 f(x) + p_2 f(y)}{p_1 + p_2} - f\left(\frac{p_1 x + p_2 y}{p_1 + p_2}\right) \leq L \tag{12}$$

is satisfied.

Lemma 5.4. *Let the function $f : [0, b] \to \mathbb{R}$ be convex. If there exist two positive constants l and L such that $bl \geq aL$ and for every $x, y \in [0, b]$ and $p_1 + p_2 > 0$ the condition (12) is satisfied, then the function $\widetilde{f} : [0, 1] \to \mathbb{R}$ is convex too.*

Proof. Under the conditions of the Lemma, for every $0 \leq x, y \leq b$ and $p_1 + p_2 > 0$ we have

$$\begin{aligned}
&\frac{p_1 \widetilde{f}(x) + p_2 \widetilde{f}(y)}{p_1 + p_2} - \widetilde{f}\left(\frac{p_1 x + p_2 y}{p_1 + p_2}\right) \\
&\qquad = b\left(\frac{p_1 f(bx) + p_2 f(by)}{p_1 + p_2} - f\left(\frac{p_1 bx + p_2 by}{p_1 + p_2}\right)\right) \\
&\qquad -a\left(\frac{p_1 f(ax) + p_2 f(ay)}{p_1 + p_2} - f\left(\frac{p_1 ax + p_2 ay}{p_1 + p_2}\right)\right) \\
&\qquad \geq bl - aL \geq 0. \ \square
\end{aligned}$$

Theorem 5.5. *Let $f : [0, b] \to \mathbb{R}$ be a continuous and convex function. If there exist two positive constants l and L such that $bl \geq aL$ and for every $x, y \in [0, b]$, $p_1 + p_2 > 0$ the condition (12) is satisfied, then it holds*

$$bf\left(\frac{b}{[2]_q}\right) - af\left(\frac{a}{[2]_q}\right) \leq I_q(f; a, b) \leq \frac{(b-a)qf(0) + bf(b) - af(a)}{[2]_q}. \tag{13}$$

Proof. According to Lemma 5.4, the function $\widetilde{f}$ is convex on $[0, 1]$. Then, using Theorem 5.2, we have

$$\widetilde{f}\left(\frac{1}{[2]_q}\right) \leq I_q(\widetilde{f}; 0, 1) \leq \frac{1}{[2]_q}\left(q\ \widetilde{f}(0) + \widetilde{f}(1)\right).$$

Applying the relation (7) we get the statement. $\square$

Corollary 5.6. *Let $f : [0, a+b] \to \mathbb{R}$ be a continuous and convex function. If there exist two positive constants l and L such that $bl \geq aL$ and for every $x, y \in [0, a+b]$, $p_1 + p_2 > 0$ the condition* (12) *is satisfied, then it holds*

$$l + f\Big(\frac{a+b}{[2]_q}\Big) \leq \frac{1}{b-a} I_q(f; a, b) \leq \frac{1}{[2]_q}\Big(qf(0) + f(a+b) + L\Big).$$

Proof. Let $p_1 = b/(b-a)$, $p_2 = -a/(b-a)$. Applying the condition (12) with $x = b/(1+q)$, $y = a/(1+q)$ on the left term and $x = a$, $y = b$ on the right term in (13), we get the statement. □

6 The other inequalities

In this section we will formulate some new inequalities for $G_q(\,\cdot\,; a, b)$, $I_q(\,\cdot\,; 0, b)$ and $R_q(\,\cdot\,; a, b)$. They will be proven only for $G_q(\,\cdot\,; a, b)$. In the way presented in the previous sections, these inequalities for the other two types follow directly. Furthermore, it seems that the corresponding inequalities for the integral $I_q(\,\cdot\,; a, b)$ defined by (2), exist and have different forms because of the previously mentioned difficulties related to estimating of the difference of series.

So, let $J_q(\,\cdot\,) = J_q(\,\cdot\,; a_{(J)}, b)$ denotes the q–integral defined by (1), (3) or (4). In the formulation and proofs of the theorems we follow the inequalities for the finite sums given in [1].

The first class are the inequalities the Cauchy-Buniakowsky-Schwarz type.

Theorem 6.1. *Let $f, g : E_{(J)} \to \mathbb{R}$ be two real functions and $\alpha, \beta > 1$ the numbers satisfying $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1$. Then the following inequalities hold:*

$$(i) \qquad \frac{1}{\alpha} J_q(|f|^\alpha) + \frac{1}{\beta} J_q(|g|^\beta) \geq \frac{1}{b - a_{(J)}} J_q(|f|) J_q(|g|),$$

$$(ii) \qquad \frac{1}{\alpha} J_q(|f|^\alpha) J_q(|g|^\alpha) + \frac{1}{\beta} J_q(|f|^\beta) J_q(|g|^\beta) \geq \Big(J_q(|fg|)\Big)^2,$$

$$(iii) \qquad \frac{1}{\alpha} J_q(|f|^\alpha) J_q(|g|^\beta) + \frac{1}{\beta} J_q(|f|^\beta) J_q(|g|^\alpha) \geq J_q(|f||g|^{\alpha-1}) J_q(|f||g|^{\beta-1}),$$

$$(iv) \qquad J_q(|f|^\alpha) J_q(|g|^\beta) \geq J_q(|fg|) J(|f|^{\alpha-1}|g|^{\beta-1}).$$

Proof. If in well-known Young inequality (see [5], pp. 381)

$$\frac{1}{\alpha} x^\alpha + \frac{1}{\beta} y^\beta \geq xy \qquad (x, y \geq 0, \quad \alpha, \beta > 1 : \ \frac{1}{\alpha} + \frac{1}{\beta} = 1),$$

we put $x = |f(bq^i)|$, $y = |g(bq^j)|$, where $i, j = 0, 1, \ldots, n-1$, we have

$$\frac{1}{\alpha} |f(bq^i)|^\alpha + \frac{1}{\beta} |g(bq^j)|^\beta \geq |f(bq^i)||g(bq^j)|, \quad i, j = 0, 1, \ldots, n-1.$$

Multiplying by q^{i+j} and summing over i and j, we obtain

$$\frac{1}{\alpha}\sum_{j=0}^{n-1} q^j \sum_{i=0}^{n-1} q^i |f(bq^i)|^\alpha + \frac{1}{\beta}\sum_{i=0}^{n-1} q^i \sum_{j=0}^{n-1} q^j |g(bq^j)|^\beta \geq \sum_{i=0}^{n-1} q^i |f(bq^i)| \sum_{j=0}^{n-1} q^j |g(bq^j)|$$

and, finally, inequality (i). The rest of inequalities can be proved in the same manner by the next choice of the parameters in Young inequality:

$$\begin{array}{llll} (ii) & x = |f(bq^j)|\ |g(bq^i)|, & y = |f(bq^i)|\ |g(bq^j)|, & \\ (iii) & x = |f(bq^j)|/|g(bq^j)|, & y = |f(bq^i)|/|g(bq^i)|, & \big(g(bq^j)\ g(bq^i) \neq 0\big), \\ (iv) & x = |f(bq^i)|/|f(bq^j)|, & y = |g(bq^i)|/|g(bq^j)|, & \big(f(bq^j)\ g(bq^j) \neq 0\big), \end{array}$$

where additional conditions about not vanishing for f and g do not have influence on final conclusion. $\square$

Theorem 6.2. *Let $f, g : E_{(J)} \to \mathbb{R}$ be two real functions and $\alpha, \beta > 1$ the numbers satisfying $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then the following inequalities hold:*

$$(i)\ \frac{1}{\alpha} J_q(|f|^\alpha) J_q(|g|^2) + \frac{1}{\beta} J_q(|f|^2) J_q(|g|^\beta) \geq J_q(|fg|) J_q(|f|^{2/\beta}|g|^{2/\alpha}),$$

$$(ii)\ \frac{1}{\alpha} J_q(|f|^2) J_q(|g|^\beta) + \frac{1}{\beta} J_q(|f|^\alpha) J_q(|g|^2) \geq J_q(|f|^{2/\alpha}|g|^{2/\beta}) J_q(|f|^{\alpha-1}|g|^{\beta-1}),$$

$$(iii)\qquad J_q(|f|^2) J_q\Big(\frac{1}{\alpha}|g|^\alpha + \frac{1}{\beta}|g|^\beta\Big) \geq J_q(|f|^{2/\alpha}|g|) J_q(|f|^{2/\beta}|g|).$$

Proof. As previous, the proof is based on Young inequality with appropriate choice of the parameters:

$$\begin{array}{llll} (i) & x = |f(bq^i)|\ |g(bq^j)|^{2/\alpha}, & y = |f(bq^j)|^{2/\beta}\ |g(bq^i)|, & \\ (ii) & x = |f(bq^i)|^{2/\alpha}/|f(bq^j)|, & y = |g(bq^i)|^{2/\beta}/|g(bq^j)| & (f(bq^j) g(bq^j) \neq 0), \\ (iii) & x = |f(bq^i)|^{2/\alpha}\ |g(bq^j)|, & y = |f(bq^j)|^{2/\beta}\ |g(bq^i)|. \ \square & \end{array}$$

The following few inequalities include the boundaries of the functions.

Theorem 6.3. *If $f, g : E_{(J)} \to \mathbb{R}$ are two positive functions and*

$$m = \min_{a \leq x \leq b} \frac{f(x)}{g(x)}, \qquad M = \max_{a \leq x \leq b} \frac{f(x)}{g(x)},$$

then the following inequalities hold:

$$(i)\qquad 0 \leq J_q(f^2) J_q(g^2) \leq \frac{(m+M)^2}{4mM} \Big(J_q(fg)\Big)^2,$$

$$(ii)\quad 0 \leq \sqrt{(J_q(f^2) J_q(g^2)} - J_q(fg) \leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}} J_q(fg),$$

$$(iii)\quad 0 \leq J_q(f^2) J_q(g^2) - \Big(J_q(fg)\Big)^2 \leq \frac{(M-m)^2}{4mM} \Big(J_q(fg)\Big)^2.$$

Proof. With respect to the definition of $G_q(\,\cdot\,; a, b)$, the inequality (i) is the immediate consequence of the Cassels inequality (see [1], pp. 72). The inequalities (ii) and (iii) can be obtained by a few transformations of (i). $\square$

Theorem 6.4. *If $f, g : E_{(J)} \to \mathbb{R}$ are two positive functions such that*

$$0 < c \le f(x) \le C < \infty, \qquad 0 < d \le g(x) \le D < \infty,$$

then the following inequalities hold:

$$(i) \qquad 0 \le J_q(f^2) J_q(g^2) \le \frac{(cd + CD)^2}{4cdCD} \Big(J_q(fg)\Big)^2,$$

$$(ii) \qquad 0 \le \sqrt{J_q(f^2) J_q(g^2)} - J_q(fg) \le \frac{(\sqrt{CD} - \sqrt{cd})^2}{2\sqrt{cdCD}} J_q(fg),$$

$$(iii) \quad 0 \le J_q(f^2) J_q(g^2) - \Big(J_q(fg)\Big)^2 \le \frac{(CD - cd)^2}{4cdCD} \Big(J_q(fg)\Big)^2.$$

Proof. Under the conditions satisfied by the functions f and g, we have

$$\frac{c}{D} \le \frac{f(x)}{g(x)} \le \frac{C}{d}.$$

Applying Theorem 6.3 we get the inequality (i) and, using it, (ii) and (iii). $\square$

Corollary 6.5. *Let $f : E_{(J)} \to \mathbb{R}$ be a positive function such that*

$$0 < c \le f(x) \le C < \infty.$$

Then the following inequality holds:

$$J_q(f^2) \le \frac{(c + C)^2}{4cC\big(b - a_{(J)}\big)} \Big(J_q(f)\Big)^2.$$

The next few inequalities are obtained via Jensen inequality for the convex functions.

Theorem 6.6. *Let $f, g : E_{(J)} \to \mathbb{R}$ be two positive functions and $p \ne 0$ a real number. Then it holds*

$$\Big(J_q(fg)\Big)^p \le \Big(J_q(f^2)\Big)^{p-1} J_q(f^{2-p} g^p), \quad \textit{for} \quad p \notin (0, 1),$$

$$\Big(J_q(fg)\Big)^p \ge \Big(J_q(f^2)\Big)^{p-1} J_q(f^{2-p} g^p), \quad \textit{for} \quad p \in (0, 1).$$

Proof. For $p \notin (0, 1)$ the function $t \mapsto t^p$ is convex. Applying the Jensen inequality for convex functions (see [5], pp.6.) we have

$$\left(\frac{\sum\limits_{k=0}^{n-1} f(bq^k) g(bq^k) q^k}{\sum\limits_{k=0}^{n-1} \big(f(bq^k)\big)^2 q^k}\right)^p \le \frac{1}{\sum\limits_{k=0}^{n-1} \big(f(bq^k)\big)^2 q^k} \sum_{k=0}^{n-1} \Big(\frac{g(bq^k)}{f(bq^k)}\Big)^p \big(f(bq^k)\big)^2 q^k,$$

i.e.,

$$\left(\sum_{k=0}^{n-1} f(bq^k)g(bq^k)q^k\right)^p \le \left(\sum_{k=0}^{n-1}\big(f(bq^k)\big)^2 q^k\right)^{p-1}\left(\sum_{k=0}^{n-1}\big(g(bq^k)\big)^p\big(f(bq^k)\big)^{2-p}q^k\right).$$

According to the definition of $G_q(\,\cdot\,; a, b)$ we get the inequality. The reverse case is obtained for $p \in (0,1)$ because of the concave function $t \mapsto t^p$. $\square$

Corollary 6.7. *Let $f : E_{(J)} \to \mathbb{R}$ be a positive function and $p \neq 0$ a real number. Then it holds*

$$\Big(J_q(f)\Big)^p \le \big(b - a_{(J)}\big)^{p-1}\; J_q(f^p),$$

for $p \notin (0,1)$, or reverse for $p \in (0,1)$.

Theorem 6.8. *If $f, g : E_{(J)} \to \mathbb{R}$ are two positive functions such that*

$$0 < m \le \frac{g(x)}{f(x)} \le M < \infty$$

and $p \neq 0$ a real number, then it holds

$$J_q(f^{2-p}g^p) + \frac{mM(M^{p-1} - m^{p-1})}{M - m} J_q(f^p) \le \frac{M^p - m^p}{M - m} J_q(fg),$$

for $p \notin (0,1)$, or reverse for $p \in (0,1)$. Especially, for $p = 2$, we have

$$J_q(g^2) + mMJ_q(f^2) \le (M + m)J_q(fg).$$

Proof. The inequality is based on the Lah-Ribarić inequality (see [5], pp. 9., [1], pp. 123). $\square$

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References

[1] S.S. Dragomir, *"A survey on Cauchy-Bunyakovsky-Schwarz type discrete inequalities"*, J. Inequalities in Pure and Applied Mathematics, **4**(3) Art. 63, 2003.

[2] H. Gauchman, *Integral Inequalities in q-Calculus*, Computers and Mathematics with Applications, vol. 47, (2004), 281–300.

[3] G. Gasper and M. Rahman, *"Basic hypergeometric series"*, Cambridge University Press, London and New York, 1990.

[4] W. Hahn, *"Lineare Geometrische Differenzengleichungen"*, 169 Berichte der Mathematisch-Statistischen Section im Forschungszentrum Graz, 1981.

[5] D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *"Classical and New Inequalities in Analysis"*, Kluwer Academic Publishers, 1993.

[6] P.M. Rajković, M.S. Stanković and S.D. Marinković, *The zeros of polynomials orthogonal with respect to q-integral on several intervals in the complex plane*, Proceedings of The Fifth International Conference on Geometry, Integrability and Quantization, 2003, Varna, Bulgaria (ed. I.M. Mladenov, A.C.Hirsshfeld), 178-188.